




---

**Contents**


---

|                             |     |
|-----------------------------|-----|
| <b>How to Use This Book</b> | iii |
|-----------------------------|-----|

---

|                         |     |
|-------------------------|-----|
| <b>Acknowledgements</b> | vii |
|-------------------------|-----|

---

|          |                                                      |          |
|----------|------------------------------------------------------|----------|
| <b>1</b> | <b>Review of Counting &amp; Probability Basics</b>   | <b>1</b> |
| 1.1      | Introduction . . . . .                               | 1        |
| 1.2      | Basic Counting Techniques . . . . .                  | 2        |
| 1.3      | Basic Probability Techniques . . . . .               | 11       |
| 1.4      | Expected Value . . . . .                             | 18       |
| 1.5      | Pascal's Triangle and the Binomial Theorem . . . . . | 20       |
| 1.6      | Summation Notation . . . . .                         | 25       |
| 1.7      | Summary . . . . .                                    | 26       |

---

|          |                              |           |
|----------|------------------------------|-----------|
| <b>2</b> | <b>Sets and Logic</b>        | <b>27</b> |
| 2.1      | Introduction . . . . .       | 27        |
| 2.2      | Sets . . . . .               | 28        |
| 2.3      | Operations on Sets . . . . . | 33        |
| 2.4      | Truth and Logic . . . . .    | 37        |
| 2.5      | Quantifiers . . . . .        | 43        |
| 2.6      | Summary . . . . .            | 46        |

---

|          |                                                        |           |
|----------|--------------------------------------------------------|-----------|
| <b>3</b> | <b>A Piece of PIE</b>                                  | <b>49</b> |
| 3.1      | Introduction . . . . .                                 | 49        |
| 3.2      | PIE With 2 Properties . . . . .                        | 49        |
| 3.3      | PIE With 3 Properties . . . . .                        | 54        |
| 3.4      | Counting Problems With PIE . . . . .                   | 58        |
| 3.5      | PIE With Many Properties . . . . .                     | 65        |
| 3.6      | Counting Items With More Than 1 of Something . . . . . | 68        |
| 3.7      | Some Harder PIE Problems . . . . .                     | 73        |
| 3.8      | Summary . . . . .                                      | 79        |

---

|          |                                                      |           |
|----------|------------------------------------------------------|-----------|
| <b>4</b> | <b>Constructive Counting and 1-1 Correspondences</b> | <b>83</b> |
| 4.1      | Introduction . . . . .                               | 83        |
| 4.2      | Some Basic Problems . . . . .                        | 84        |
| 4.3      | Harder Constructive Counting Problems . . . . .      | 86        |
| 4.4      | 1-1 Correspondence Basics . . . . .                  | 94        |
| 4.5      | More Complicated 1-1 Correspondences . . . . .       | 99        |
| 4.6      | Clever 1-1 Correspondences . . . . .                 | 105       |
| 4.7      | Summary . . . . .                                    | 114       |

---

|          |                                             |            |
|----------|---------------------------------------------|------------|
| <b>5</b> | <b>The Pigeonhole Principle</b>             | <b>118</b> |
| 5.1      | Introduction . . . . .                      | 118        |
| 5.2      | It's Just Common Sense! . . . . .           | 118        |
| 5.3      | Basic Pigeonhole Problems . . . . .         | 119        |
| 5.4      | More Advanced Pigeonhole Problems . . . . . | 122        |
| 5.5      | Summary . . . . .                           | 127        |

---

|          |                                               |            |
|----------|-----------------------------------------------|------------|
| <b>6</b> | <b>Constructive Expectation</b>               | <b>130</b> |
| 6.1      | Introduction . . . . .                        | 130        |
| 6.2      | Basic Examples . . . . .                      | 131        |
| 6.3      | Summing Expectations Constructively . . . . . | 135        |
| 6.4★     | A Coat With Many Patches (Reprise) . . . . .  | 140        |
| 6.5      | Summary . . . . .                             | 142        |

---

|          |                                                  |            |
|----------|--------------------------------------------------|------------|
| <b>7</b> | <b>Distributions</b>                             | <b>145</b> |
| 7.1      | Introduction . . . . .                           | 145        |
| 7.2      | Basic Distributions . . . . .                    | 146        |
| 7.3      | Distributions With Extra Conditions . . . . .    | 149        |
| 7.4      | More Complicated Distribution Problems . . . . . | 154        |
| 7.5      | Summary . . . . .                                | 158        |

---

|          |                               |            |
|----------|-------------------------------|------------|
| <b>8</b> | <b>Mathematical Induction</b> | <b>161</b> |
|----------|-------------------------------|------------|

---

|          |                                               |            |
|----------|-----------------------------------------------|------------|
| <b>9</b> | <b>Fibonacci Numbers</b>                      | <b>172</b> |
| 9.1      | Introduction . . . . .                        | 172        |
| 9.2      | A Motivating Problem . . . . .                | 172        |
| 9.3      | Some Fibonacci Problems . . . . .             | 175        |
| 9.4      | A Formula for the Fibonacci Numbers . . . . . | 183        |
| 9.5      | Summary . . . . .                             | 189        |

---

|           |                                              |            |
|-----------|----------------------------------------------|------------|
| <b>10</b> | <b>Recursion</b>                             | <b>192</b> |
| 10.1      | Introduction . . . . .                       | 192        |
| 10.2      | Examples of Recursions . . . . .             | 193        |
| 10.3      | Linear Recurrences . . . . .                 | 200        |
| 10.4      | A Hard Recursion Problem . . . . .           | 204        |
| 10.5      | Problems Involving Catalan Numbers . . . . . | 206        |
| 10.6      | Formulas for the Catalan Numbers . . . . .   | 216        |
| 10.7      | Summary . . . . .                            | 222        |

---

|           |                                                     |            |
|-----------|-----------------------------------------------------|------------|
| <b>11</b> | <b>Conditional Probability</b>                      | <b>227</b> |
| 11.1      | Introduction . . . . .                              | 227        |
| 11.2      | Basic Examples of Conditional Probability . . . . . | 227        |
| 11.3      | Some Definitions and Notation . . . . .             | 230        |
| 11.4      | Harder Examples . . . . .                           | 232        |
| 11.5      | Let's Make a Deal! . . . . .                        | 238        |
| 11.6      | Summary . . . . .                                   | 241        |

---

---

|           |                                 |            |
|-----------|---------------------------------|------------|
| <b>12</b> | <b>Combinatorial Identities</b> | <b>243</b> |
| 12.1      | Introduction . . . . .          | 243        |
| 12.2      | Basic Identities . . . . .      | 243        |
| 12.3      | More Identities . . . . .       | 254        |
| 12.4      | Summary . . . . .               | 261        |

---

|           |                                           |            |
|-----------|-------------------------------------------|------------|
| <b>13</b> | <b>Events With States</b>                 | <b>265</b> |
| 13.1      | Introduction . . . . .                    | 265        |
| 13.2      | State Diagrams and Random Walks . . . . . | 266        |
| 13.3      | Events With Infinite States . . . . .     | 277        |
| 13.4      | Two-player Strategy Games . . . . .       | 284        |
| 13.5      | Summary . . . . .                         | 294        |

---

|           |                                                             |            |
|-----------|-------------------------------------------------------------|------------|
| <b>14</b> | <b>Generating Functions</b>                                 | <b>298</b> |
| 14.1      | Introduction . . . . .                                      | 298        |
| 14.2      | Basic examples of Generating Functions . . . . .            | 299        |
| 14.3      | The Binomial Theorem (as a Generating Function) . . . . .   | 305        |
| 14.4      | Distributions (as Generating Functions) . . . . .           | 307        |
| 14.5      | The Generating Function for Partitions . . . . .            | 314        |
| 14.6★     | The Generating Function for the Fibonacci Numbers . . . . . | 325        |
| 14.7      | Summary . . . . .                                           | 327        |

---

|           |                                          |            |
|-----------|------------------------------------------|------------|
| <b>15</b> | <b>Graph Theory</b>                      | <b>331</b> |
| 15.1      | Introduction . . . . .                   | 331        |
| 15.2      | Definitions . . . . .                    | 332        |
| 15.3      | Basic Properties of Graphs . . . . .     | 334        |
| 15.4      | Cycles and Paths . . . . .               | 342        |
| 15.5      | Planar Graphs . . . . .                  | 349        |
| 15.6      | Eulerian and Hamiltonian Paths . . . . . | 354        |
| 15.7      | Summary . . . . .                        | 360        |

---

|           |                           |            |
|-----------|---------------------------|------------|
| <b>16</b> | <b>Challenge Problems</b> | <b>363</b> |
|-----------|---------------------------|------------|

---

---

## CONTENTS

---

|                            |     |
|----------------------------|-----|
| References                 | 366 |
| Hints to Selected Problems | 367 |
| Index                      | 383 |