Beast Academy 5
Chapter 1: 3D Solids

Sequence: • • • BA3, Chapter 3 Perimeter & Area ▸ BA3, Chapter 12 Area ▸ BA4, Chapter 1 Shapes ▸ BA5, Chapter 1 This Chapter

Students must be comfortable finding the areas of rectangles and triangles.

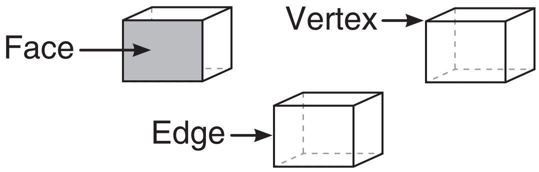
Overview

This chapter introduces three-dimensional solids and their surface area and volume. We encourage the use of manipulatives in this chapter. Just holding and unfolding a cereal box can help students visualize three dimensional objects and understand the concepts in this chapter.

It is fine for students to use a reference sheet to organize new terms and definitions.

Basics

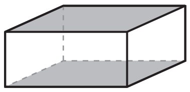
A **polyhedron** is a 3D solid with flat sides called **faces** and straight **edges** that meet at corners called **vertices**. We explore patterns in the relationships between faces, vertices, and edges.



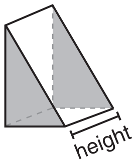
Prisms and Pyramids

A **prism** has two congruent (same size, same shape) faces that are parallel. These congruent faces are called its **bases**. All of the prisms in this chapter are right prisms, meaning the bases are always connected by rectangles.

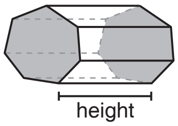
It is important to emphasize that **"base" doesn't always mean the bottom face**. In a rectangular prism, any pair of opposite faces can be used as the bases. For other prisms, the bases are the faces that are not rectangles. Prisms are named by the shape of their bases. A prism's height is the distance between its bases.



Rectangular prism

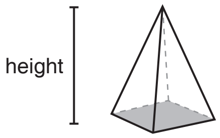


Triangular prism

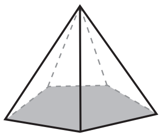


Heptagonal prism

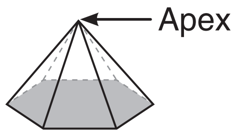
A **pyramid** has one base. All of the other faces are triangles that meet at a single point called its **apex**. Like prisms, pyramids are named by the shape of their base.



Square Pyramid



Pentagonal Pyramid



Hexagonal Pyramid

A student should be able to figure out how many faces, vertices, and edges a prism or pyramid has given the shape of its base by visualizing how all of its parts are connected.

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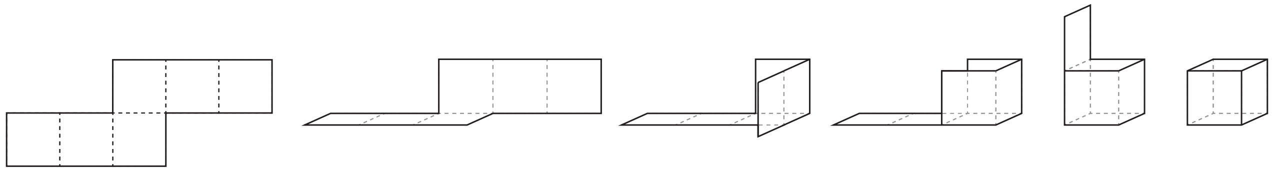
Chapter 1: 3D Solids

Nets

A net is a 2-dimensional shape that can be folded to make the surface of a 3-dimensional solid. Playing with the nets of various solids can help students improve their spatial reasoning skills.

Playing with the nets of polyhedra will also help students find their surface areas later. Printable nets for all 5 Platonic solids are at BeastAcademy.com/resources/printables.

Once students have played with physical nets, they can practice visualizing nets being folded mentally to solve a variety of cube net puzzles.

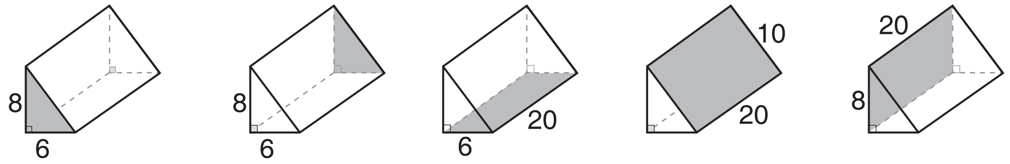


Surface Area

The surface area of a polyhedron is the sum of all the areas of its faces.

Encourage students to stay organized. “How many faces are there all together? Are any of them the same? Can you list them all? What is the area of the front face? The back face? The top?”

Students can move on to triangular prisms and even more complicated solids where organization becomes even more important. It helps for students to sketch individual faces.

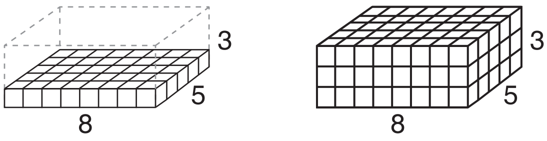


24 + 24 + 120 + 200 + 160 = 528 (square units)

Volume

Volume is the number of cubic units in a 3-dimensional solid. We focus on prism volume. Guide students to find a formula they can use to find the volume of any prism.

The area of the base of a prism tells us how many cubes it will take to cover its base. The height of the prism tells us how many layers of cubes it will take to fill the prism.



It takes 5×8 = 40 cubes to cover the area of the rectangular base. It takes 3 layers to fill the prism. So, the volume of the prism is 3×40 = 120 cubic units.

It takes (4×5) ÷ 2 = 10 cubes to cover the area of the triangular base. It takes 3 layers to fill the prism. So, the volume of the prism is 3×10 = 30 cubic units.

Students can find the volume of *any* prism similarly by multiplying the area of its base by its height.

Beast Academy 5

Chapter 2: Integers

Sequence: BA4, Chapter 9 Integers → BA5, Chapter 2 This Chapter

Students **must** be able to add and subtract positive and negative integers fluently before learning to multiply and divide with negatives. Students without a solid grasp of addition and subtraction often misapply the multiplication and division rules to addition and subtraction problems.

Overview

This chapter focuses on multiplying and dividing with positive and negative integers. Students are often given the rules without understanding why they work. Guide students to discover these rules on their own using patterns and examples. Students who understand the rules for multiplication and division are far less likely to misapply them.

Multiplying Positive and Negative Integers

Encourage students to create rules for multiplying positive and negative integers as they discover patterns.

Positive Times Negative

We write the product of a positive and a negative as repeated addition. For example, 4×(-5) is the same as adding four -5's. Adding -5's gives us a number that is "more negative." The same works for multiplying any positive times a negative. The result is always negative.

$$\begin{aligned} 4\times(-5) &= (-5)+(-5)+(-5)+(-5) \\ &= -20 \end{aligned}$$

Negative Times Positive

Since multiplication is commutative (the order of the numbers you are multiplying doesn't matter), we can use the same rule we found above. For example, since -5×4 equals 4×(-5), we know -5×4 = -20. The result of multiplying a negative times a positive is also always negative.

$$\begin{aligned} -5\times 4 &= 4\times(-5) \\ &= (-5)+(-5)+(-5)+(-5) \\ &= -20 \end{aligned}$$

Negative Times Negative

This rule is easiest to discover by looking at a pattern of products like the one on the right. Students can complete patterns of products to see that the result of multiplying two negatives is always positive.

$$\begin{array}{rcll} -3\times 3 &=& -9 & \\ & & & +3 \\ -3\times 2 &=& -6 & \\ & & & +3 \\ -3\times 1 &=& -3 & \\ & & & +3 \\ -3\times 0 &=& 0 & \\ -3\times(-1) &=& \_\_ & \\ -3\times(-2) &=& \_\_ & \end{array} \longrightarrow \begin{array}{rcll} -3\times 3 &=& -9 & \\ -3\times 2 &=& -6 & \\ -3\times 1 &=& -3 & \\ -3\times 0 &=& 0 & \\ & & & +3 \\ -3\times(-1) &=& \mathbf{3} & \\ & & & +3 \\ -3\times(-2) &=& \mathbf{6} & \end{array}$$

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Chapter 2: Integers

Longer Products

Help students discover how to quickly tell whether a product is positive or negative.

In products with more than two terms, two negatives can always be paired to make a positive.

If there are an *even* number of negatives in the product, all of the negatives can be paired to make positives, so the product is positive.

If there are an *odd* number of negatives, there will be one negative that cannot be paired, so the product is negative.

Dividing Positive and Negative Integers

Students can use the relationship between multiplication and division to find the rules for division.

	Examples:
Positive Divided by Negative	
Since $(-)\times(-)=(+)$, we know $(+)\div(-)=(-)$.	$-7\times(-4) = 28$, so $28\div(-4) = -7$.
Negative Divided by Positive	
Since $(-)\times(+)=(-)$, we know $(-)\div(+)=(-)$.	$-7\times4 = -28$, so $-28\div4 = -7$.
Negative Divided by Negative	
Since $(+)\times(-)=(-)$, we know $(-)\div(-)=(+)$.	$7\times(-4) = -28$, so $-28\div(-4) = 7$.

Opposites

A negative in front of an expression means to take its opposite. Taking the opposite of an expression is the same as multiplying it by -1.

Later on, this concept will be useful for students when distributing a negative.

$$\begin{aligned} &-(-7+5)\\ =\ &-(-2)\\ =\ &2 \end{aligned}$$

– or –

$$\begin{aligned} &-(-7+5)\\ =\ &-1\times(-7+5)\\ =\ &-1\times(-2)\\ =\ &2 \end{aligned}$$

Exponents

There is a frustrating notation issue with exponents. The expression -5^2 is ambiguous. It's not obvious whether -5^2 means $(-5)^2=(-5)\times(-5)$, or $-(5^2)=-(25)$.

The rule is that -5^2 means $-(5^2)$, which is -25.

This is easy to mess up, especially since the wrong interpretation often gives the right answer.

It's good for students to get exposure to this rule early on, as it comes up often in algebra.

It's best to use parentheses to make sure your notation is not ambiguous, using $-(5^2)$ and $(-5)^2$.

-2^3 and $(-2)^3$ mean different things, but are equal.

$$-2^3 = -(2^3) = -8$$

$$(-2)^3=(-2)\times(-2)\times(-2)=-8$$

-2^2 and $(-2)^2$ mean different things, and are not equal.

$$-2^2 = -(2^2) = -4$$

$$(-2)^2=(-2)\times(-2)=4$$

Beast Academy 5
Chapter 3: Expressions & Equations

Sequence:
BA2, Chapter 5 Expressions
BA3, Chapter 7 Variables
BA5, Chapter 3 This Chapter

Students should be comfortable with the order of operations, using variables, and negative integers.

Overview

This chapter helps students transition from arithmetic to prealgebra. The primary goal is to help students become fluent in simplifying expressions and solving equations.

New Notation

We introduce several new ways to write multiplication. We avoid using the × symbol, which looks like the variable x. For example, $2 \cdot 7$ means 2×7, $4(5-7)$ means $4 \times (5-7)$, and $2x$ means 2 times x. We also use the fraction bar instead of ÷ to write most division.

New Vocabulary

In a math expression, a **term** is a number, a variable, or a product of numbers and variables.

For example, the expression $3x^2+3xy+7$ has three terms: $3x^2$, $3xy$, and 7. If two terms have the exact same variables, we call them **like terms**. For example, $3n$ and $7n$ are like terms, and so are $11ab$ and $-12ab$.

The **coefficient** of a term is the number part. The coefficient of $3n$ is 3, and the coefficient of $-12ab$ is -12. If a term like xy doesn't have a number written, its coefficient is 1 (since $xy = 1xy$).

Knowing these words helps us talk about math. If there were no word for "coefficient," we'd have to say, "the number in front of a product of variables" which is even worse than learning a word like coefficient.

Simplifying Expressions

We can often **simplify** expressions by **combining like terms**—adding the stuff that's the same. For example, $7c+11c$ can be simplified. Adding 7 c's plus 11 c's gives us 18 c's: $7c+11c=18c$.

For more complicated expressions, we can begin by writing all of the subtraction as addition. That way, since we can add terms in any order, we can rearrange the expression to get the like terms together. Then, we simplify as shown below on the left.

Once students get the hang of it, they can combine like terms without rewriting everything. It helps to think of subtraction as adding the opposite. In other words, think of "$-6a$" as adding $-6a$. This means that if we keep the + and − signs "glued" to the numbers we're adding or subtracting, we can add them in any order. So, to simplify, we just circle like terms (with their + and − signs) and add them.

Simplify the expression
$7-6a-2+5b+3a-b$.

$$\begin{aligned} 7-6a-2+5b+3a-b &= 7+(-6a)+(-2)+5b+3a+(-b) \\ &= \underbrace{-6a+3a}_{} + \underbrace{5b+(-b)}_{} + \underbrace{7+(-2)}_{} \\ &= -3a + 4b + 5 \end{aligned}$$

—or—

$$\bigcirc\!\!\!\!\!7 \; \bigcirc\!\!\!\!\!\!\!\!-6a \; \bigcirc\!\!\!\!\!\!\!-2 \; \bigcirc\!\!\!\!\!\!\!\!+5b \; \bigcirc\!\!\!\!\!\!\!\!+3a \; \bigcirc\!\!\!\!\!\!\!-b$$

$$5 \qquad -3a \qquad +4b$$

$$= -3a + 4b + 5$$

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Chapter 3: Expressions & Equations

Solving Equations

Solving an equation means figuring out the value of the variable(s). To solve $a+4=13$, most students will "see" that a is 9. But as equations get more complicated, students need better ways to solve them. This usually involves working backwards. We introduce this with word problems. Try this one:

If I take my favorite number, double it, take away 7, then divide the result by 5, I get 9.
What's my favorite number?

To figure out the favorite number, we undo everything that was done to it. Before you divided by 5, you had $9\times5=45$. Before you took away 7, you had $45+7=52$. Before you doubled, you had $52\div2=26$. So, your favorite number is 26.

Most equations can be solved like this. For example, the word problem above can be written as an equation, with f representing the favorite number:

$$\frac{2f-7}{5}=9.$$

Ask students to describe what happens to f to give us 9.

As in the word problem, we start with a number (f), multiply it by 2, subtract 7, divide by 5 and get 9. We solve the equation just like the word problem—by working backwards. To undo dividing by 5, we multiply by 5. To undo subtracting 7, we add 7. To undo multiplying by 2, we divide by 2.

Solve the equation $\frac{2f-7}{5}=9$.

$2f-7$ divided by 5 equals 9.	$\frac{2f-7}{5} = 9.$
Before we divided $2f-7$ by 5, we had $9\times5=45$. So, $2f-7=45$.	$2f-7 = 45$
Before we subtracted 7 from $2f$, we had $45+7=52$. So, $2f=52$.	$2f = 52$
And before we multiplied by 2, we had $52\div2=26$. So, $f=26$.	$f = 26$

Solving equations is usually taught as doing the same thing to both sides of an equation to get the variable by itself. As long as we do the same thing to both sides, they will stay equal:

1. Multiply both sides by 5.

$$5\bullet\left(\frac{2f-7}{5}\right)=9\bullet5.$$
$$2f-7 = 45$$

2. Add 7 to both sides.

$$\begin{array}{rcr} 2f-7 & = & 45 \\ +7 & & +7 \\ \hline 2f & = & 52 \end{array}$$

3. Divide both sides by 2.

$$\frac{2f}{2} = \frac{52}{2}$$
$$f = 26$$

This is great, and we can apply this idea to any equation. But, students often have trouble figuring out what steps to take, or in what order. For example, it's common for students to first add 7 to both sides to 'cancel' the 7.

Sadly, $\frac{2f-7}{5}+7$ gives us $\frac{2f-7}{5}+\frac{35}{5}=\frac{2f+28}{5}$, and we're not any closer to getting f by itself.

Undoing what happened to f in reverse order helps students solve equations efficiently.

Beast Academy 5
Chapter 4: Statistics

Sequence: BA5, Chapter 4 This Chapter *(This is the only statistics chapter in the BA curriculum.)*

Overview

Statistics is the study of data. In this chapter, we introduce four basic statistical measures: average (mean), median, mode, and range. We focus on the average and explain how these measures can be used to describe a set of data.

Median

The median is the middle number of an ordered list. There are the same number of numbers above and below the median.

If there is an even number of numbers in the list, the median is the number halfway between the two middle numbers.

What is the median of this list?
72 19 17 29 33 55 54 41

First, we list the numbers in order:

17 19 29 33 41 54 55 72

The median is the number exactly between 33 and 41, which is 37.

Average (Mean)

Our approach to average involves **equal sharing**.

If all of the amounts in a list are shared equally among a group, the average is the amount each person gets. Explain the concept of the average as equal sharing and encourage students to figure out their own ways of finding the average.

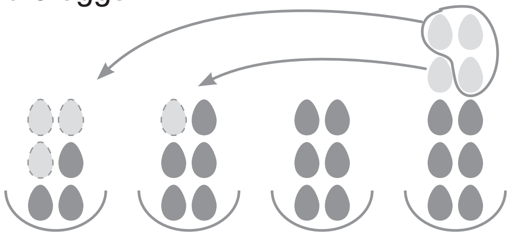
Try distributing items unevenly to a group of students and ask them to figure out how to share the items equally (the total number of items should probably be divisible by the number of people).

Students may use several methods like the example given on the right. Encourage these strategies. Equal sharing approaches like these will help students understand the more sophisticated strategies later in the chapter.

All students should eventually arrive at the traditional strategy of finding the average by adding all of the numbers, and then dividing by how many numbers there are.

Quan has four bowls, containing 3, 5, 6, and 10 eggs. If Quan wants to arrange the eggs so each bowl holds the same number, how many eggs will be in each bowl?

Quan could take four eggs from the bowl that has 10 eggs and put one in the bowl that holds 5, and three in the bowl that holds 3. Each bowl will then hold 6 eggs.



Or, since there are a total of $3+5+6+10=24$ eggs to put in 4 bowls, there will be $24\div4=6$ eggs in each bowl.

Beast Academy 5

Chapter 4: Statistics

Balancing Around the Average

The numbers in a set of data always **balance** around the average. The numbers *above* the average balance the numbers *below* the average.

For example, if the numbers below the average in a list are *less than* the average by a total of 26, then the numbers above the average must be *more than* the average by a total of 26.

This gives us a way to solve problems involving average without adding all of the numbers in a list. It also gives students a way of thinking about average without using a specific process or formula.

The average of the eight numbers in the list below is 40. What is the missing number?

30 33 34 37 41 45 45 $\underline{?}$

We compare each number on the list to the average (40). The four numbers on the left are *less than* the average by a total of 26, so the four numbers on the right must be *more than* the average by a total of 26.

$$30\quad 33\quad 34\quad 37\quad \overset{\text{Avg}}{(40)}\quad 41\quad 45\quad 45\quad \underline{?}$$
$$\underbrace{-10\quad -7\quad -6\quad -3}_{-26}\qquad\qquad \underbrace{+1\quad +5\quad +5\quad +?}_{+26}$$

So, the missing number must be 15 more than the average, which is $40+15=55$.

$$30\quad 33\quad 34\quad 37\quad \overset{\text{Avg}}{(40)}\quad 41\quad 45\quad 45\quad \underline{55}$$
$$\underbrace{-10\quad -7\quad -6\quad -3}_{-26}\qquad\qquad \underbrace{+1\quad +5\quad +5\quad +15}_{+26}$$

Averaging Averages

Students can find the combined average of two lists of numbers that have different averages.

In the example to the right, the average weight of all 16 fruits is *not* $(5+9)\div 2=7$ ounces. Is the average closer to the weight of a grapefruit, or to the weight of an apple? Why?

Help students discover that they can't just average the two averages, and that the average weight must be closer to the weight of the apple since there are more of them.

Twelve apples weigh an average of 5 ounces each. Four grapefruits weigh an average of 9 ounces each. What is the average weight of all 16 fruits?

The twelve apples weigh a total of $12\times 5=60$ ounces, and the four grapefruits weigh a total of $4\times 9=36$ ounces.

The total weight is $60+36=96$ ounces, so the average weight of all 16 fruits is $96\div 16=6$ ounces.

How Many?

Encourage students to try various strategies to tackle novel problems that can't be solved just by applying a formula. Problems where students have to find the total size of a list are great for balance strategies.

Tanya has 1 quarter (25¢) and some dimes (10¢). The average value of all her coins is 11¢. How many dimes does she have?

Since the quarter is 14¢ above the average, and each dime is just 1¢ below the average, Tanya must have 14 dimes to "balance" the quarter.

Range, Mode, and Data Display

Two very different lists of numbers can have the same average and median. Range (the difference between the smallest and largest numbers) and mode (the number that appears most) give a clearer picture of a data set.

Beast Academy 5
Chapter 5: Factors & Multiples

Sequence: BA4, Chapter 7 Factors → BA2, Chapter 4 This Chapter → BA5, Chapter 11 Square Roots → BA5, Chapter 12 Exponents

Before beginning this chapter, students must be able to identify a number's factor pairs and find its prime factorization using a factor tree.

Overview

This chapter emphasizes prime factorization and the relationship between factors and multiples. Mastery of these concepts is necessary for students to understand the GCF and LCM relationships introduced in this chapter.

Students will apply these skills to solve many types of problems, including factoring algebraic expressions. For example, students will factor $3x$ from $3x^2-6x$ to get $3x(x-2)$.

Factors and Multiples

It is very important that students understand the relationship between factors and multiples.

One number is a factor of a second number if the first number's prime factorization is included in the second number's prime factorization. This also means the second number is a multiple of the first.

For example, we can look at the prime factorization of 2,660 to see that 28, 95, 190, and 14 are all factors of 2,660 (and that 2,660 is a multiple of all four).

We can also see that $8=2\times2\times2$ is *not* a factor of 2,660, since there are only two 2's in the prime factorization of 2,660.

Is $28 = 2^2 \cdot 7$ a factor of $2{,}660 = 2^2 \cdot 5 \cdot 7 \cdot 19$?

$$\begin{aligned} 2{,}660 &= 2 \cdot 2 \cdot 5 \cdot 7 \cdot 19 \\ &= (2 \cdot 2 \cdot 7) \cdot (5 \cdot 19) \\ &= 28 \cdot 95 \end{aligned}$$

Is $190 = 2 \cdot 5 \cdot 19$ a factor of $2{,}660 = 2^2 \cdot 5 \cdot 7 \cdot 19$?

$$\begin{aligned} 2{,}660 &= 2 \cdot 2 \cdot 5 \cdot 7 \cdot 19 \\ &= (2 \cdot 5 \cdot 19) \cdot (2 \cdot 7) \\ &= 190 \cdot 14 \end{aligned}$$

Introducing GCF (Greatest Common Factor) and LCM (Least Common Multiple)

The terms GCF and LCM are easily confused. We recommend students initially learn to find GCF and LCM without using prime factorization. This will help them solidify the meaning of GCF and LCM.

The Greatest Common Factor of 180 and 234 is 18.

180		234
$1 \cdot 180$	$6 \cdot 30$	$1 \cdot 234$
$2 \cdot 90$	$9 \cdot 20$	$2 \cdot 117$
$3 \cdot 60$	$10 \cdot \textcircled{18}$	$3 \cdot 78$
$4 \cdot 45$	$12 \cdot 15$	$6 \cdot 39$
$5 \cdot 36$		$9 \cdot 26$
		$13 \cdot \textcircled{18}$

The Least Common Multiple of 90 and 120 is 360.

Multiples of 90: 90, 180, 270, $\textcircled{360}$, 450, ...

Multiples of 120: 120, 240, $\textcircled{360}$, 480, ...

Beast Academy 5

Chapter 5: Factors & Multiples

GCF (Using prime factorizations)

We can find the GCF of 180 and 234 using their prime factorizations.

180 = 2 • 2 • 3 • 3 • 5   234 = 2 • 3 • 3 • 13

The goal is to help students discover that the greatest common factor of a pair or set of numbers is the product of **all** their common prime factors.

Guide students with questions like, “Is 6 a factor of both 180 and 234? How can you tell? Is there a larger number that is a factor of both 180 and 234?”

For example, students should see that since 180 and 234 both have two 3’s in their prime factorizations, 3 • 3 = 9 is a factor of both. But it’s not the **greatest** factor they have in common.

180 and 234 both have at least one 2, and at least two 3’s. They don’t have any other factors in common. So, their GCF is 2 • 3 • 3 = **18**.

LCM (Using prime factorizations)

Similarly, we can use prime factorizations to find the **LCM** of 180 and 234:

180 = 2 • 2 • 3 • 3 • 5   234 = 2 • 3 • 3 • 13

The goal is to help students realize that the LCM is the smallest number that includes the prime factorizations of both numbers.

Ask questions like “Does a multiple of 180 and 234 have any 2’s in its prime factorization? How many?” How many 3’s? How many 7’s? Is 2 • 2 • 3 • 5 • 13 a multiple of both 180 and 234? How can you tell?”

Every multiple of 180 must have at least two 2’s, two 3’s, and one 5 in its prime factorization.

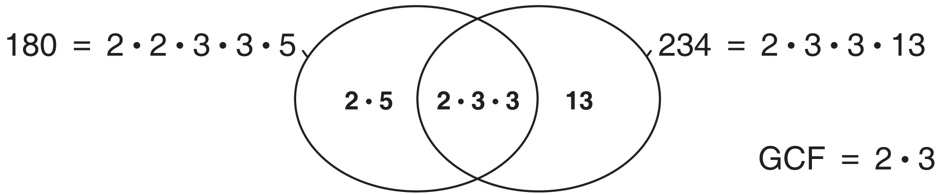
Every multiple of 234 must have at least one 2, two 3’s, and one 13 in its prime factorization.

So, a multiple of ***both*** numbers must have at least two 2’s, two 3’s, one 5, and one 13 in its prime factorization. The least common multiple is the number with only these prime factors:

$2^2 \cdot 3^2 \cdot 5 \cdot 13 =$ **2,340**.

Venn Diagrams

Venn diagrams are used to help students visualize these relationships.



GCF = 2 • 3 • 3 = 18

LCM = 2 • 2 • 3 • 3 • 5 • 13 = 2,340

Factorials

The sections on Factorials are optional. We introduce factorials to give students another avenue for practice with prime factorizations.

Beast Academy 5
Chapter 6: Fractions

Sequence: ••• BA4, Chapter 10 Fractions ▶ BA4, Chapter 11 Decimals ▶ **BA5, Chapter 6 This Chapter** ▶ BA5, Chapter 8 Ratios •••

It is very important that students begin this chapter with a firm understanding of fractions as presented in the fraction chapters before it. Before beginning, students should be able to fluently:

- convert between fractions and mixed numbers,
- label and order fractions and mixed numbers on the number line,
- add and subtract fractions with like denominators, and
- multiply and divide whole numbers by fractions and mixed numbers.

Overview

In this chapter, we tie everything students have learned about fractions together. By the end, students should be able to add, subtract, multiply, or divide using fractions and mixed numbers.

As always, aim for conceptual understanding as described in each section below. Avoid teaching formulas and processes without understanding.

Addition and Subtraction (Unlike Denominators)

Tie new concepts to ones students already understand.

Adding only makes sense if the things we're adding are the same in some way. For example, 3 apples plus 5 oranges equals 8 *fruits*, because they are both fruits. To add 2 meters plus 40 centimeters, you would use a common unit to get 2.4 meters, or 240 centimeters. Similarly, to add fractions like fifths and thirds, you need to make them the same by converting both to fifteenths.

Students who learn to use a rule without understanding why it works have trouble remembering when and how to apply it. For example, after learning to use a common denominator for addition, some students may try to use a common denominator for multiplication (which works, but is inefficient).

Students who struggle with multiplication facts may have difficulty finding common denominators. Give problems that have small denominators with obvious common factors at first, and guide students to look for the smallest possible denominator.

It's ok if students use *any* common denominator and simplify afterwards, but encourage students to look for the smallest common denominator when adding fractions. "Is there a smaller denominator that would have worked?"

Add $\frac{3}{10}+\frac{2}{5}$.

First, we convert $\frac{2}{5}$ into tenths.

$$\frac{2}{5} \underset{\cdot 2}{\overset{\cdot 2}{=}} \frac{4}{10}$$

Then, we add:

$$\frac{3}{10}+\frac{2}{5}=\frac{3}{10}+\frac{4}{10}=\frac{7}{10}.$$

Add $\frac{5}{12}+\frac{3}{8}$.

First, we convert $\frac{5}{12}$ and $\frac{3}{8}$ into 24ths.

$$\frac{5}{12} \underset{\cdot 2}{\overset{\cdot 2}{=}} \frac{10}{24} \qquad \frac{3}{8} \underset{\cdot 3}{\overset{\cdot 3}{=}} \frac{9}{24}$$

Then, we add:

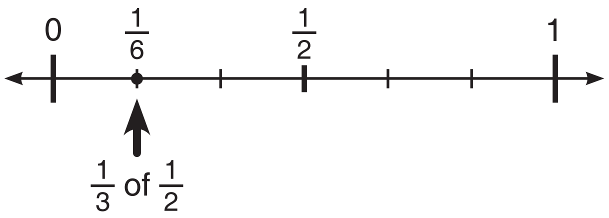
$$\frac{5}{12}+\frac{3}{8}=\frac{10}{24}+\frac{9}{24}=\frac{19}{24}.$$

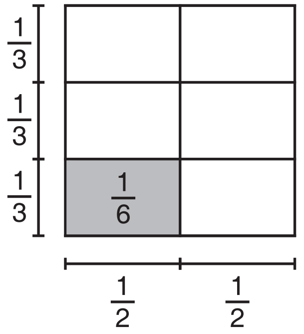
Beast Academy 5

Chapter 6: Fractions

Multiplying Fractions

Multiplying $\frac{1}{3}\cdot\frac{1}{2}$ means finding $\frac{1}{3}$ of $\frac{1}{2}$. We can show this on the number line and in a unit square. Both models point to the idea that to multiply two unit fractions, we multiply their denominators.





$$\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{3\cdot 2}=\frac{1}{6}.$$

Students can use this along with what they know about fractions and multiplication to multiply *any* two fractions as shown on the right.

Multiply $\frac{2}{5}\cdot\frac{3}{7}$.

$$\begin{aligned}\frac{2}{5}\cdot\frac{3}{7} &= 2\cdot\frac{1}{5}\cdot 3\cdot\frac{1}{7}\\ &= (2\cdot 3)\cdot\left(\frac{1}{5}\cdot\frac{1}{7}\right)\\ &= 6\cdot\left(\frac{1}{35}\right)\\ &= \frac{6}{35}\end{aligned}$$

Students will notice that all we're doing is multiplying the numerators to get the numerator, and the denominators to get the denominator.

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}.$$

So, why didn't we just say that in the first place!?

Guiding students to understand a rule helps them remember the rule and apply it correctly, and gives students tools that will be useful for other skills (like the simplifying that comes next).

Simplifying

We can use similar strategies to simplify fractions by "cancelling" common factors. Simplifying fractions before we multiply can make computations a lot easier.

Multiply $\frac{2}{3}\cdot\frac{6}{7}$.

Showing all of the steps:

$$\frac{2}{3}\cdot\frac{6}{7}=\frac{2\cdot 6}{3\cdot 7}=\frac{6\cdot 2}{3\cdot 7}=\frac{6}{3}\cdot\frac{2}{7}=2\cdot\frac{2}{7}=\frac{4}{7}$$

After some practice:

$$\frac{2}{{}_1\cancel{3}}\cdot\frac{\cancel{6}^{2}}{7}=\frac{2\cdot 2}{1\cdot 7}=\frac{4}{7}$$

Division

Simple examples give students intuition for how division by a fraction works. To solve $3\div\frac{1}{4}$, students can reason that there are 4 fourths in 1, so there are 3 • 4 fourths in 3. So, $3\div\frac{1}{4}=3\cdot 4=12$.

Students learned in Beast Academy 4D that to divide by any number, we multiply by its reciprocal. A number's reciprocal is the number you multiply it by to get 1.

Since $\frac{a}{b}\cdot\frac{b}{a}=\frac{ab}{ba}=1$, the reciprocal of any fraction $\frac{a}{b}$ is $\frac{b}{a}$. So, to divide by $\frac{a}{b}$, we multiply by $\frac{b}{a}$.

Beast Academy 5

Chapter 7: Sequences

Sequence: BA3, Chapter 7 Variables → BA5, Chapter 3 Expressions & Eq → BA5, Chapter 4 Statistics → BA5, Chapter 7 Place Value

Before beginning this chapter, students must be comfortable manipulating expressions and equations and should understand the concept of average. It may also help if students have used expressions to describe patterns as shown in the variables chapter of BA3.

Overview

Finding patterns is an important part of mathematics. This chapter introduces patterns and arithmetic sequences (sequences where we add the same number over and over again), and helps students describe them using algebraic expressions.

It is important not to take a formulaic approach to solving sequence problems, where students plug numbers into mysterious formulas they don't understand. Arithmetic sequences are just skip-counting patterns, and students should be able to reason their way through problems.

Many of the puzzles in this chapter are very challenging (even by BA standards), but the math is approachable and the puzzles are a lot of fun for students looking for a good mental workout.

Basics

Students learn some new vocabulary. A **sequence** is a list of **terms**, usually numbers, that follow a pattern. Each term has a **position** in the sequence (1st, 2nd, 3rd, etc.). If a sequence has ellipses (...) at the end, it continues forever. Sequences that go on forever are called **infinite sequences**.

Arithmetic Sequences

In an arithmetic sequence, the same amount is always added to get from one term to the next (that number is negative if the numbers in the sequence are getting smaller). The amount that is added is called the **common difference**.

Encourage students to reason their way through finding unknown terms in an arithmetic sequence.

What is the 50th term in the sequence below?
7, 13, 19, 25, 31, 37, ...

Encourage This

+6 +6 +6 +6 +6
7, 13, 19, 25, 31, 37, ...

We're counting by 6's, starting at 7.
To get the 2nd term we add one 6.
To get the 3rd term, we add two 6's.
To get the 4th term, we add three 6's.
We want the 50th term, so we need to add forty-nine 6's.
So, the 50th term is $7+(49\times6) = 7+294 = 301$.

Not This

The formula for the n^{th} term of an arithmetic sequence is:

$$a_n = a_1+(n-1)d$$

n^{th} term
1st term
common difference
term position

Plugging in, we get $a_{50} = 7+(50-1)6 = 301$.

Beast Academy 5

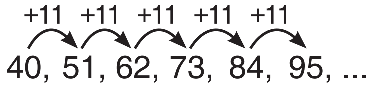
Chapter 7: Sequences

The n^{th} Term

Students can write expressions that will help them find *any* term of a sequence (usually called the "n^{th} term," where n is the term's position). Students may come up with different expressions. That's great! Encourage them to always check their expressions to make sure they work.

Below, we show two ways to write expressions for the n^{th} term in an arithmetic sequence.

Write an expression that represents the n^{th}
term of the sequence:
40, 51, 62, 73, 84, 95, ...

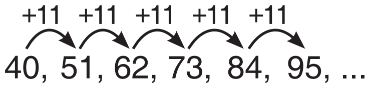


We're adding 11's, starting at 40.

The 2nd term is 40 plus one 11.
The 3rd term is 40 plus two 11's.
The 4th term is 40 plus three 11's.

We always add one less 11 than the term's position.

So, the n^{th} term is 40 plus $n-1$ elevens, which is $40+11(n-1)$.



We're adding 11's.

The 1st term is $29+11$.
The 2nd term is 29 plus two 11's.
The 3rd term is 29 plus three 11's.

So, the n^{th} term is 29 plus n elevens, which is $29+11n$.

Students may find many expressions that work. With a little bit of work, it's usually not hard to show students that all of the expressions that work are equal. For example, in the expression on the left, we can distribute the 11 in $40+11(n-1)$ to get $40+11n-11$. Then, we can simplify to get $29+11n$, which is the expression we got on the right.

Students also write expressions for other sequences; for example, sequences where the n^{th} term is a power like (n^2) or $(-1)^n$.

Arithmetic Series

An arithmetic series is a sum of terms in an arithmetic sequence. For example, $2+4+6+8$ is an arithmetic series that equals 20. Avoid giving students formulas. Instead, encourage students reason through finding the sum as shown in one of the two methods below.

Add: $50+52+54+56+58+60+62+64+68+70$.

We can add the list of numbers to itself, with one copy written backwards.

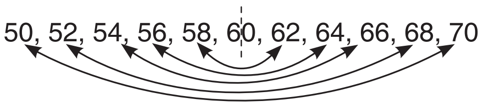
$$\begin{array}{ccccccccccccccccccccc}
50 & + & 52 & + & 54 & + & 56 & + & 58 & + & 60 & + & 62 & + & 64 & + & 66 & + & 68 & + & 70 \\
\underline{70} & \underline{+} & \underline{68} & \underline{+} & \underline{66} & \underline{+} & \underline{64} & \underline{+} & \underline{62} & \underline{+} & \underline{60} & \underline{+} & \underline{58} & \underline{+} & \underline{56} & \underline{+} & \underline{54} & \underline{+} & \underline{52} & \underline{+} & \underline{50} \\
120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120 & + & 120
\end{array}$$

We have 11 pairs of numbers that sum to 120.

So, the sum of *two* copies of the list is $11\times120=1{,}320$.

So, the sum of *one* copy of the list is $1{,}320\div2=660$.

We can find the average of the terms (which, for a series, equals the median), then multiply by the number of terms.



The 11 terms have an average of 60.
So, their sum is $60\times11=660$.

Beast Academy 5
Chapter 8: Ratios & Rates

Sequence: BA5, Chapter 6 Fractions → BA5, Chapter 8 This Chapter → BA5, Chapter 9 Decimals → BA5, Chapter 10 Percents

Before beginning this chapter, students must have a solid understanding of fractions.

Overview

A ratio describes a relationship between two quantities. For example, the ratio of water to flour in a bread recipe is 3-to-5. Ratios are usually written as two values separated by a colon; the ratio of water to flour is written 3:5.

A ratio doesn't tell us how much of each quantity there is, only how quantities are related. For example, you could get a 3:5 ratio by using 30 cups of water and 50 cups of flour, or by using 1.5 cups of flour 2.5 cups of flour. Each has 3 parts water for every 5 parts flour.

In some ways, ratios work a lot like fractions. So, we often use fraction notation to write and compare ratios. However, unlike fractions, ratios are not numbers on the number line. Ratios simply describe relationships. For example, we don't add or subtract ratios.

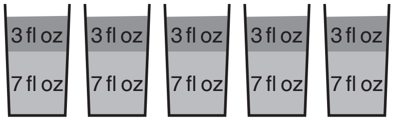
Using Ratios

It is often useful to consider how the parts in a ratio relate to the whole amount. For example, in the water-to-flour ratio of 3:5 above, we can think of a total mixture that is 3 parts water and 5 parts flour for a total of 8 parts; 3 of those 8 parts are water and 5 of the 8 parts are flour.

Below, we show several ways to reason through the same ratio problem.

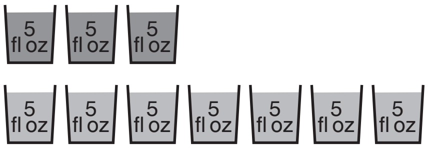
The ratio of soda to juice in a fruit punch recipe is 3:7. How many ounces of each will you need to make a 50-ounce fruit punch mix?

For every 3 fluid ounces of soda in the fruit punch, there are 7 fluid ounces of juice. So, in every 10-ounce serving of punch, there are 3 ounces of soda and 7 ounces of juice. There are five 10-ounce servings in a 50-ounce mix. So, we need $\underline{3} \cdot 5 = 15$ fl oz of soda and $\underline{7} \cdot 5 = 35$ fl oz of juice.



— or —

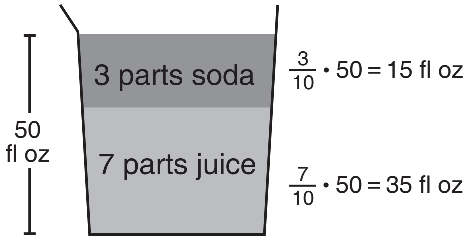
For every 3 parts soda, there are 7 parts juice, for a total of $3+7=10$ parts. If we divide 50 fluid ounces of punch into 10 equal parts, each part is 5 fluid ounces. So, the 3 parts of soda is $\underline{3} \cdot 5 = 15$ fl oz and the 7 parts juice is $\underline{7} \cdot 5 = 35$ fl oz.



— or —

In a punch that is 3 parts soda and 7 parts juice, 3 of the 10 parts are soda and 7 of the 10 parts are juice. In other words, $\frac{3}{10}$ of the punch is soda, and $\frac{7}{10}$ of the punch is juice.

So, to make 50 fluid ounces of punch, we need $\frac{3}{10} \cdot 50 = 15$ fl oz of soda and $\frac{7}{10} \cdot 50 = 35$ fl oz of juice.



Beast Academy 5

Chapter 8: Ratios & Rates

Proportional Reasoning

Two ratios are equivalent if they have the same simplest form. For example, all of the ratios below can be simplified to 3:5. This is easiest to see when they are written as fractions.

$$540:900 = 54:90 = 27:45 = 9:15 = 3:5 = 21:35 = 105:175$$

$$\begin{tikzcd}[column sep=small]
\frac{540}{900} \arrow[rr, bend left, "\div 10"] \arrow[rr, bend right, "\div 10"'] & = & \frac{54}{90} \arrow[rr, bend left, "\div 2"] \arrow[rr, bend right, "\div 2"'] & = & \frac{27}{45} \arrow[rr, bend left, "\div 3"] \arrow[rr, bend right, "\div 3"'] & = & \frac{9}{15} \arrow[rr, bend left, "\div 3"] \arrow[rr, bend right, "\div 3"'] & = & \frac{3}{5} \arrow[rr, bend left, "\times 7"] \arrow[rr, bend right, "\times 7"'] & = & \frac{21}{35} \arrow[rr, bend left, "\times 5"] \arrow[rr, bend right, "\times 5"'] & = & \frac{105}{175}
\end{tikzcd}$$

A **proportion** is an equation that shows two ratios are equal. Encourage students to come up with their own methods for solving proportions by "scaling" both parts of the ratio up or down by the same factor. Below are some examples of ways kids might solve some basic proportions.

$$\begin{tikzcd}[column sep=small]
\frac{6}{10} \arrow[rr, bend left, "\times 3"] \arrow[rr, bend right, "\times 3"'] & = & \frac{x}{30}
\end{tikzcd}$$

$$x = 18$$

$$\begin{tikzcd}[column sep=small]
\frac{15}{16} \arrow[rr, bend left, "\div 3"] \arrow[rr, bend right, "\div 3"'] & = & \frac{5}{z}
\end{tikzcd}$$

$$z = \frac{16}{3}$$

$$\frac{8}{20} = \frac{10}{t}$$

Since $\frac{8}{20} = \frac{2}{5}$,
and $\frac{2}{5} = \frac{10}{25}$,
$t = 25$.

$$\begin{tikzcd}[column sep=small]
\frac{18}{n} & = & \frac{12}{14} \arrow[ll, bend right, "\times 1.5"'] \arrow[ll, bend left, "\times 1.5"]
\end{tikzcd}$$

$n = 21$.

Once students understand methods like the ones above, they can learn to compare ratios using a common denominator. This can lead them to cross multiplication as a method of solving proportions.

Rates

A **rate** is a special kind of ratio that compares quantities that have different units. For example, if a farm has 8 sheep for every 5 acres, the ratio of sheep to acres (8:5) is a rate. Most of the time we use rates, we use **unit rates**. A unit rate describes how much of a quantity there is for *one unit* of another quantity. Unit rates usually use the word "per," as in "30 miles per hour."

In the 8:5 sheep-to-acres ratio above, there are $\frac{8}{5} = 1.6$ sheep per acre, which is a unit rate.

One advantage of using unit rates is that it makes comparison easy. For example, comparing which is the better deal—$3.60 for a 5-pound bag, or $6 for an 8-pound bag—we can convert both to unit rates. The 5-pound bag is $0.72/lb, and the 8-pound bag is $0.75/lb. So, the 5-pound bag is a (slightly) better deal.

$$\begin{tikzcd}[column sep=small]
\frac{\$3.60}{5\text{ lbs}} \arrow[rr, bend left, "\div 5"] \arrow[rr, bend right, "\div 5"'] & = & \frac{\$0.72}{1\text{lb}}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
\frac{\$6}{8\text{ lbs}} \arrow[rr, bend left, "\div 8"] \arrow[rr, bend right, "\div 8"'] & = & \frac{\$0.75}{1\text{lb}}
\end{tikzcd}$$

Beast Academy 5

Chapter 9: Decimals

Sequence: BA5, Chapter 6 Fractions → BA5, Chapter 8 Ratios → BA5, Chapter 9 This Chapter → BA5, Chapter 10 Percents

Before beginning this chapter, students must have a solid understanding of place value and must be able to add and subtract decimals, which is covered in BA4, Chapter 11.

Overview

This chapter focuses on multiplying decimals and on converting between fractions and decimals.

Multiplying Decimals by Powers of 10 (10, 0.1, 10,000, 0.001, etc.)

Each place value is 10 times the place value to its right.

Multiplying a number by 10 moves each digit to the next-larger place value. This is the same as moving its decimal point one place to the right. For example, 3.45×10 = 34.5.

Compute 3.45×10.

$$3.45\times 10 = 34.5$$

Multiplying a number by 0.1 moves each digit to the next-smaller place value. This is the same as moving its decimal point one place to the left. For example, 98.7×0.1 = 9.87.

Compute 98.7×0.1.

$$98.7\times 0.1 = 9.87$$

Using these two facts, we can multiply by any power of 10.

Since 10,000 = 10×10×10×10, multiplying by 10,000 is the same as multiplying by 10 four times. So, the decimal point moves four places.

Compute 56.7×10,000.

$$56.7\times 10{,}000 = 567\underbrace{,000}_{4}$$

Since 0.001 = 0.1×0.1×0.1, multiplying by 0.001 is the same as multiplying by 0.1 three times, so the decimal point moves 3 places.

Compute 45.6×0.001.

$$45.6\times 0.001 = 0.\underbrace{045}_{3}6$$

It's important not to focus on memorizing rules for whether the decimal moves left or right. Instead, students should consider what makes sense. Questions like, "Does the number get bigger or smaller?" help students avoid memorization.

Multiplying Decimals

Once students understand how to multiply by powers of 10, encourage students to use those skills to help them multiply any two decimals. Guide students to look for patterns and write their own "rules".

Relate multiplying decimals to multiplying whole numbers, with the added difficulty of figuring out where the decimal point goes. Start simple. "What's 3×0.1? What's 3×0.2? How about 0.3×0.2?" The goal is to reach more difficult computations like the example below.

With a little practice and experimentation, students should be able to find and describe patterns for multiplying numbers with decimals.

$$\begin{aligned} 0.\underbrace{03}_{\substack{2\\ \text{digits}}}\times 0.\underbrace{009}_{\substack{3\\ \text{digits}}} &= (3\times 0.01)\times(9\times 0.001)\\ &= (3\times 9)\times(0.01\times 0.001)\\ &= 27\times 0.00001\\ &= \mathbf{0.\underbrace{00027}_{2+3=5\text{ digits}}}. \end{aligned}$$

Students should use multiplication with powers of 10 to discover and write "rules" like:

When multiplying decimals, we can count the total number of digits to the right of the decimal point to figure out where to place the decimal point in their product.

Multiplying Decimals (continued)

Watch out for trailing zeros!

Challenge students to explain computations like $0.2\times0.5=0.1$ and 0.25×0.008, where the “rules” they have come up with may seem hard to apply because of trailing zeros.

$$\begin{aligned} 0.\underbrace{25}_{\substack{2\\\text{digits}}}\times 0.\underbrace{008}_{\substack{3\\\text{digits}}} &= (25\times0.01)\times(8\times0.001)\\ &= 200\times0.00001\\ &= \mathbf{0.\underbrace{00200}_{2+3=5\text{ digits}}}.\end{aligned}$$

Since $25\times8=200$, we include two zeros when we place the decimal point. Then, we can remove the trailing zeros. $0.25\times0.008=\mathbf{0.002}$.

Even better, since we know $0.25=\frac{1}{4}$, we see that one fourth of 0.008 is 0.002.

Encourage mental computation and estimation.

Estimating and mental computation with decimals can be hard, but always be on the lookout for quick ways to estimate and compute. In the example above, for instance, students who recognize 0.25 as one-fourth can see that 0.25×0.008 means “one-fourth of 0.008,” which is 0.002.

Converting Fractions to Decimals

Terminating Decimals

Fractions that can be written with a denominator that is a power of 10 are easiest to write as decimals, using what we know about place value.

These will always be terminating decimals (decimals that don’t continue forever).

Write $\frac{1}{25}$ and $\frac{7}{8}$ as decimals.

$$\frac{1}{25} \overset{\times4}{\underset{\times4}{=}} \frac{4}{100} = \mathbf{0.04} \qquad \frac{7}{8} \overset{\times125}{\underset{\times125}{=}} \frac{875}{1{,}000} = \mathbf{0.875}$$

Repeating Decimals

Fractions that cannot be written with a denominator that is a power of 10 are usually easiest to convert to decimals using long division.

This is no fun at all, and always gives a repeating decimal. We write a bar over the repeating part of the decimal.

Write $\frac{15}{37}$ as a decimal.

$$\begin{aligned}\frac{15}{37} &= 15\div37\\ &= 0.405405\ldots\\ &= \mathbf{0.\overline{405}}\end{aligned}$$

$$\begin{array}{r} 0.405405\ldots\\ 37\overline{)\,15.0}\\ -14.8\\ \hline 0.200\\ -0.185\\ \hline 0.0150\\ -0.0148\\ \hline \ldots\end{array}$$

Other Strategies

This is a fun but very difficult (optional) section that explores relationships between fractions and decimals that give clever conversion strategies.

For example, students can use the fact that ninths give repeating decimals to write 90ths as repeating decimals.

Use the fact that $\frac{5}{9}=0.\overline{5}$ to write $\frac{5}{90}$ as a decimal.

$$\frac{5}{90} = \frac{5}{9}\times\frac{1}{10} = 0.\overline{5}\times0.1 = \mathbf{0.0\overline{5}}.$$

Beast Academy 5

Chapter 10: Percents

Sequence: BA5, Chapter 6 Fractions → BA5, Chapter 8 Ratios → BA5, Chapter 9 Decimals → **BA5, Chapter 10 This Chapter**

Students must have a solid understanding of fractions, ratios, and decimals before beginning.

Overview

A percent is a **ratio** that means “for every 100.” If we say “25% of the monsters are purple,” it means that for every 100 monsters, 25 are purple. It does not mean that there are exactly 100 monsters and that 25 are purple. It could mean 1 of 4 monsters are purple, or 15 of 60.

Percents give us a standard way to write ratios in a way that makes them easy to compare.

For example, it is easier to see that 37.5% > 35% than $\frac{3}{8} > \frac{7}{20}$.

percent

For every 100.

Per means “for every” as in 30 miles <u>per</u> gallon.

Cent means “100” as in 100 years in a <u>cent</u>ury or 100 <u>cent</u>s in a dollar.

Conversions

We can convert percents to fractions and decimals (and back).

Percent to Fraction

Since a percent is a ratio of a number to 100, we can write the number (without the % sign) over 100, then simplify if needed.

$$36\% = \frac{36}{100} \underset{\div 4}{\overset{\div 4}{=}} \frac{9}{25}$$

Fraction to Percent

Convert the fraction so that its denominator is 100.

$$\frac{3}{8} \underset{\times 12.5}{\overset{\times 12.5}{=}} \frac{37.5}{100} = 37.5\%$$

Converting Between Percents and Decimals

The rules for converting between percents and decimals are often taught as rules to memorize that involve moving the decimal point two places one direction or the other. Students who try to memorize these rules without understanding why they work often get mixed up and have a lot of trouble with large and small percents and decimals (like 950% or 0.009).

Instead, students can use what they know about fractions to convert percents to decimals ($950\% = \frac{950}{100} = 9.5$, for example) and decimals to percents ($0.009 = \frac{9}{1{,}000} = \frac{0.9}{100} = 0.9\%$).

Then, they can recognize patterns in how the decimal point moves. Converting a decimal to a percent, the decimal point moves 2 places to the right. Converting a percent to a decimal, the decimal point moves 2 places to the left. Students can then use a common conversion like 25% = 0.25 to help them remember which way to move the decimal.

Percent of a Number

We can find a percent of a number by converting the percent to a fraction or decimal and multiplying.

What number is 20% of 80?

20% = 0.2.
So, 20% of 80 is $0.2 \times 80 = 16.0$.

–or–

$20\% = \frac{1}{5}$.
So, 20% of 80 is $\frac{1}{5} \times 80 = 16$.

Beast Academy 5

Chapter 10: Percents

Equations

Many percent problems can be solved by setting up an equation and solving for the variable.

Proportions

A proportion is an equation that shows two ratios are equal.

In the example on the right, 15% means 15 out of every 100 fish are purple. As a fraction, the ratio of purple fish to total fish is $\frac{15}{100}$.

We are given that 24 of the fish (f) are purple. So, the ratio of purple fish to total fish is $\frac{24}{f}$.

Since these two fractions are the same ratio, they are equal. We set up a proportion: $\frac{15}{100}=\frac{24}{f}$.

We solve for f using the strategies learned in BA 5 Chapter 8.

15% of the fish in a tank are purple. If there are 24 purple fish, how many total fish are in the tank?

We let f stand for the total number of fish and write a proportion:

$$\frac{24}{f}=\frac{15}{100}\begin{array}{l}\longleftarrow \text{purple fish}\\ \longleftarrow \text{total fish}\end{array}$$

Solving for f, we get $f=160$.

Other Equations

We can "translate" many sentences into equations to help us solve problems involving percents.

In the example below, we are trying to find a number, n, so that 35% of n is 28.
We write "35% of n is 28" as an equation and solve for n.

35% of what number is 28?

$35\% = \frac{35}{100} = \frac{7}{20}$.
So, $\frac{7}{20}$ of n is 28.

As an equation, we have

$$\frac{7}{20}\cdot n=28.$$

To solve for n, we multiply both sides of the equation by $\frac{20}{7}$.

$$\frac{7}{20}\cdot n = 28$$
$$\frac{\cancel{20}}{\cancel{7}}\cdot\frac{\cancel{7}}{\cancel{20}}\cdot n = \frac{20}{\cancel{7}}\cdot\overset{4}{\cancel{28}}$$
$$n = \mathbf{80}.$$

We check our answer.
35% of 80 is
$\frac{7}{20}\cdot 80=28$. ✓

We could also have used decimals to write and solve the equation $0.35n = 28$.

Percent Change

Percents are often used to express how a value changes. For example, a monster's height might increase by 30%, or price might decrease by 25%. There are two common ways to compute a percent change, as shown below.

What number is 75% more than 36?

Method 1:

To find 75% ***more than*** 36, we can find 75% of 36, then add that amount to 36.

75% is $\frac{75}{100}=\frac{3}{4}$. So, 75% of 36 is $\frac{3}{4}\cdot 36 = 27$.
Therefore, 75% more than 36 is $36+27 = \mathbf{63}$.

Method 2:

75% ***more than*** 36 is 100% of 36 plus 75% of 36.

So, 75% more than 36 is 175% of 36.

175% is $\frac{175}{100}=\frac{7}{4}$. So, 175% of 36 is $\frac{7}{4}\cdot 36 = \mathbf{63}$.

Beast Academy 5
Chapter 11: Square Roots

AoPS Beast Academy
Art of Problem Solving

Sequence: BA4, Chapter 7 Factors → BA5, Chapter 5 Factors & Multiples → BA5, Chapter 11 This Chapter → BA5, Chapter 12 Exponents

Students must be very comfortable with factoring before beginning this chapter.

Overview

This chapter introduces students to square roots and the Pythagorean theorem.

Basics

The square root of a number is the nonnegative value we square to get the number. For example, since we square 4 to get 16, the square root of 16 is 4.

To avoid confusion, the square root of a number is never negative. So, even though 4^2 and $(-4)^2$ both equal 16, the only square root of 16 is 4.

The square root symbol, $\sqrt{}$, called a radical, is a grouping symbol. We compute everything under the radical before finding the square root.

Familiar square roots can help us find square roots of other numbers. For example, knowing $\sqrt{64}=8$ makes it easier to find $\sqrt{6{,}400}=80$ and $\sqrt{0.0064}=0.08$.

Careful! It is easy to make mistakes when finding square roots this way. For example, $\sqrt{250}$ is ***not*** 50, since $50^2=2{,}500$, not 250. The square root of 250 is actually a little less than 16 (since $16^2=256$).

We can find square roots of fractions. For example, since $\frac{2}{7}\cdot\frac{2}{7}=\frac{4}{49}$, we have $\sqrt{\frac{4}{49}}=\frac{2}{7}$.

By definition, if we square the square root of a number, we get the number. For example, $\left(\sqrt{9}\right)^2=9$ and $\left(\sqrt{15}\right)^2=15$.

$$\sqrt{16}=4$$

$$\sqrt{9+16}=\sqrt{25}=5$$

$$\sqrt{0.01}=0.1$$

$$\sqrt{0.0036}=0.06$$

$$\sqrt{\frac{4}{49}}=\frac{2}{7}$$

$$\left(\sqrt{5}\right)^2=5$$

Comparing

For *positive* numbers, the bigger the number, the bigger its square. 53 is more than 52, so 53^2 is more than 52^2.

We can compare any two positive numbers by comparing their squares.

For example, to figure out whether $\sqrt{250}$ is more or less than 15, we can square both:

$$\left(\sqrt{250}\right)^2=250\text{, and }15^2=225.$$

Since $\left(\sqrt{250}\right)^2$ is greater than 15^2, we know that $\sqrt{250}$ is greater than 15.

Which consecutive integers is $3\sqrt{11}$ between?

First, we square $3\sqrt{11}$.

$$\begin{aligned}\left(3\sqrt{11}\right)^2 &= 3\sqrt{11}\cdot 3\sqrt{11}\\ &= (3\cdot 3)\cdot\left(\sqrt{11}\cdot\sqrt{11}\right)\\ &= 9\cdot 11\\ &= 99.\end{aligned}$$

Since $\left(3\sqrt{11}\right)^2=99$ is between $9^2=81$ and $10^2=100$, $3\sqrt{11}$ is between 9 and 10.

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Chapter 11: Square Roots

Tricky Square Roots (Factoring)

Factoring can help us find many tricky square roots. The goal is to write the expression under the radical as a perfect square so we can find its square root. This is a great opportunity for factoring review and practice.

Compute $\sqrt{15 \cdot 375}$.

We can factor 15 and 375, then rearrange the factors to make a perfect square:

$$\begin{aligned}\sqrt{15 \cdot 375} &= \sqrt{(3 \cdot 5) \cdot (3 \cdot 5 \cdot 5 \cdot 5)} \\ &= \sqrt{(3 \cdot 5 \cdot 5) \cdot (3 \cdot 5 \cdot 5)} \\ &= \sqrt{(3 \cdot 5 \cdot 5)^2} \\ &= 3 \cdot 5 \cdot 5 \\ &= \mathbf{75}.\end{aligned}$$

Compute $\sqrt{\frac{12}{11} \cdot \frac{27}{275}}$.

$$\begin{aligned}\sqrt{\frac{12}{11} \cdot \frac{27}{275}} &= \sqrt{\frac{2 \cdot 2 \cdot 3}{11} \cdot \frac{3 \cdot 3 \cdot 3}{5 \cdot 5 \cdot 11}} \\ &= \sqrt{\frac{(2 \cdot 3 \cdot 3) \cdot (2 \cdot 3 \cdot 3)}{(5 \cdot 11) \cdot (5 \cdot 11)}} \\ &= \sqrt{\left(\frac{2 \cdot 3 \cdot 3}{5 \cdot 11}\right) \cdot \left(\frac{2 \cdot 3 \cdot 3}{5 \cdot 11}\right)} \\ &= \sqrt{\left(\frac{2 \cdot 3 \cdot 3}{5 \cdot 11}\right)^2} \\ &= \frac{2 \cdot 3 \cdot 3}{5 \cdot 11} \\ &= \mathbf{\frac{18}{55}}\end{aligned}$$

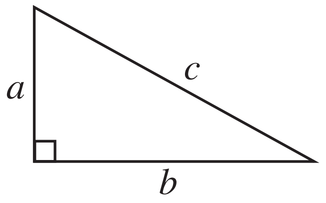
The Pythagorean Theorem

The Pythagorean theorem describes a relationship between the two short sides of a right triangle, called its ***legs***, and its longest side, called its ***hypotenuse***.

For any right triangle, the sum of the squares of its legs is equal to the square of its hypotenuse. If we know two side lengths in a right triangle, we can use the Pythagorean Theorem to find its third side.

Right triangles are everywhere! We don't always see them, though. Sometimes we need to add a line to a diagram. For example, the diagonal of a rectangle splits it into two right triangles where the diagonal is the hypotenuse.

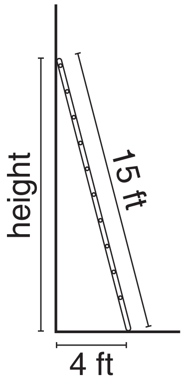
The Theorem:



In any right triangle with short sides (legs) a and b, and long side (hypotenuse) c, we have:

$$a^2 + b^2 = c^2$$

A 15-foot ladder leans against a wall. Its base is 4 feet from the wall. How high is the top of the ladder?



We use h for the height to the top of the ladder. We write and solve an equation:

$$\begin{aligned}4^2 + h^2 &= 15^2 \\ 16 + h^2 &= 225 \\ h^2 &= 209\end{aligned}$$

Since $h^2 = 209$, the height of the top of the ladder is $\sqrt{209}$ feet, which is about 14.5 feet.

There is also a negative solution to $h^2 = 209$, but distances cannot be negative, so we can ignore it.

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Chapter 12: Exponents

Sequence: BA4, Chapter 7 Factors ▸ BA5, Chapter 5 Factors & Multiples ▸ BA5, Chapter 11 Square Roots ▸ BA5, Chapter 12 This Chapter

Students must be very comfortable with factoring and fraction operations before starting this chapter.

Overview

The goal in this chapter is to help students use what they've learned about exponents to discover some rules and apply them. Even though all of these exponent rules follow from a basic understanding of multiplication, division, and exponents, this is a challenging chapter that includes some concepts that are often not taught until algebra in many curricula.

Exponent rules should not be given as a list of formulas that must be memorized. Students can discover these rules for exponents on their own. Once the rules make sense, they are much easier to recall and apply.

Multiplying and Dividing Powers

Multiplication

Students can write-out (expand) powers using multiplication to help them discover the rule for multiplying two powers that have the same base.

Write $11^{12} \times 11^{13}$ as a power of 11.

Encourage This

$$11^{12} \cdot 11^{13} = \underbrace{(11 \cdot 11 \cdot \cdots \cdot 11)}_{\text{12 elevens}} \cdot \underbrace{(11 \cdot 11 \cdot \cdots \cdot 11)}_{\text{13 elevens}}$$

Multiplying $11^{12} \cdot 11^{13}$ gives a product of $12+13=25$ elevens. So, $11^{12} \cdot 11^{13} = 11^{25}$.

Not This

Use the formula $a^m \cdot a^n = a^{m+n}$.

When we multiply two powers that have the same base, we add their exponents.

So, $11^{12} \cdot 11^{13} = 11^{12+13} = 11^{25}$.

Once they are comfortable multiplying powers as shown on the left, the add-the-exponents formula used on the right is obvious and easy to remember. Students should only use the formula once they understand it and can explain why it works.

Division

Students can use what they know about multiplication, division, and fractions to find a rule for dividing powers that have the same base. For example, to divide $7^5 \div 7^3$, we can write the division as a fraction. Then, we write the powers as multiplication and cancel:

$$7^5 \div 7^3 = \frac{7^5}{7^3} = \frac{\cancel{7} \cdot \cancel{7} \cdot \cancel{7} \cdot 7 \cdot 7}{\cancel{7} \cdot \cancel{7} \cdot \cancel{7}} = 7 \cdot 7 = 7^2.$$

Once students are comfortable with dividing powers this way, they can recognize that to divide powers with the same base, they can subtract their exponents:

Help students discover: $a^m \div a^n = a^{m-n}$. As a fraction, $\frac{a^m}{a^n} = a^{m-n}$.

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Chapter 12: Exponents

Zero and Negative Exponents

Have students explore patterns with powers of a number to infer what it means when an exponent is zero or negative.

For example, on the right we list the powers of 5. To get the next *larger* power of 5, we multiply by 5. To get the next *smaller* power of 5, we divide by 5.

So, to get 5^0, we can divide 5^1 by 5. This gives us $5^0 = 5 \div 5 = 1$. Similarly, to get 5^{-1}, we divide 5^0 by 5. Since $5^0 = 1$, this gives us $5^{-1} = 1 \div 5 = \frac{1}{5}$.

We can continue this way to get $5^{-2} = \frac{1}{5} \div 5 = \frac{1}{25} = \frac{1}{5^2}$. Next, $5^{-3} = \frac{1}{25} \div 5 = \frac{1}{125} = \frac{1}{5^3}$.

$$\begin{tikzcd} 5^3 = 125 \arrow[d, bend left, "\div 5"] \\ 5^2 = 25 \arrow[d, bend left, "\div 5"] \\ 5^1 = 5 \arrow[d, bend left, "\div 5"] \\ 5^0 = 1 \arrow[d, bend left, "\div 5"] \\ 5^{-1} = \frac{1}{5} \arrow[d, bend left, "\div 5"] \\ 5^{-2} = \frac{1}{25} \end{tikzcd}$$

This pattern continues and works for any nonzero base. We have two rules:

Help students discover:

Any number raised to the power of 0 is 1.

$$a^0 = 1.$$

If a is not zero, a^{-n} is the reciprocal of a^n.

$$a^{-n} = \frac{1}{a^n}.$$

We can use the rules for multiplying and dividing exponents to support both rules above.

For example, $7^4 \div 7^4 = 7^{4-4}$, which is 7^0. Since we're dividing a number by itself, $7^4 \div 7^4 = 1$. So, $7^0 = 1$.

Similarly, $7^3 \div 7^5 = 7^{3-5}$, which is 7^{-2}. We know $\frac{7^3}{7^5} = \frac{\cancel{7}\cdot\cancel{7}\cdot\cancel{7}}{7\cdot 7\cdot\cancel{7}\cdot\cancel{7}\cdot\cancel{7}} = \frac{1}{7\cdot 7} = \frac{1}{7^2}$. So, $7^{-2} = \frac{1}{7^2}$. Yay math!

Raising a Power to a Power

Students can reason through a problem like the one below to help them discover the rule for raising a power to a power.

Write $(5^4)^3$ as a power of 5.

5^4 is the product of 4 copies of 5:

$$5^4 = 5\cdot 5\cdot 5\cdot 5.$$

$(5^4)^3$ is the product of 3 copies of 5^4:

$$(5^4)^3 = 5^4\cdot 5^4\cdot 5^4.$$

So, $(5^4)^3$ is 3 copies of $5\cdot 5\cdot 5\cdot 5$:

$$\begin{aligned}(5^4)^3 &= (5\cdot 5\cdot 5\cdot 5)\cdot(5\cdot 5\cdot 5\cdot 5)\cdot(5\cdot 5\cdot 5\cdot 5).\\ &= 5^{12}.\end{aligned}$$

For a problem like $(5^7)^9$, students can use the same reasoning without writing everything out. 9 groups of 7 fives is a total of $7\cdot 9 = 63$ fives. So, $(5^7)^9 = 5^{7\cdot 9} = 5^{63}$. Let students find the rule below on their own.

Help students discover: When raising a power to a power, multiply the exponents and keep the same base.

$$(a^m)^n = a^{m\cdot n}$$

Same Exponent

What happens when you multiply or divide two bases with the same exponent?

Is $2^4\cdot 3^4 = (2\cdot 3)^4$? Is $12^3 \div 4^3 = (12 \div 4)^3$?

Write out the expressions and see! It may help to use fractions for the second one.

Memorizing rules like these leads to a lot of confusion. All of the operations are easy to mix up. Expand, rearrange, and cancel. Figure out what works and why. Once the rules make sense, no memorization is needed.